malaysia olympiad on astronomy and astrophysics

Theoretical Round

Good Luck and Clear Skies!

General Instructions

1. The **Theoretical Round** consists of 6 questions on astronomical theory.
   The allocated time for this round is **180 minutes**.
2. The round will be announced to start by the invigilator (Camp Facilitator).
   Please **DO NOT** turn over this page before the start of this round.
3. A countdown timer would be shown on the screen. There are no restrictions to the time budget for each question.
4. There is a total of **140** marks allocated. The marks attributed to each question is marked below the problem statement. The problems are not sorted by difficulty, and many subparts are independent of previous results, **attempt as many as possible**!
5. You will be provided working sheets for your solutions. On **EACH** working sheet, write down your **Name**, **IC Number** and **Question Number**. All answers that are to be evaluated must be written on the sheets provided. You are advised to use separated pages for separate questions.
6. **Cross out** sections that you do not want to be evaluated.
7. Use as many mathematical expressions to help the graders better understand your solutions. The graders may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
8. You are not allowed to leave your working desk without permission. If you need any assistance (malfunctioning calculator, restroom visits, insufficient or missing sheets, etc.), please put up your hand to signal the invigilator.
9. The round would end once the countdown timer rings. At the end of the round, you must stop writing immediately. Sort and put your working sheets in one stack. Put papers you do not want to be graded in another stack. You are allowed to keep this question paper.
10. Please remain seated until your papers are collected.

Table of Constants

Mass ($M_\oplus$)	6×10^{24} kg	Earth
Radius ($R_\oplus$)	6.4×10^{6} m	
Obliquity (ε)	$23.5°$	
Mass ($M_\odot$)	2×10^{30} kg	Sun
Radius ($R_\odot$)	6.96×10^{8} m	
Luminosity ($L_\odot$)	3.82×10^{26} W	
Surface Temperature ($T_\odot$)	5778 K	
Apparent V-band magnitude ($m_{\odot,V}$)	-26.8 mag	
Absolute V-band magnitude ($\mathcal{M}_\odot$)	4.83 mag	
Gravitational constant (G)	6.67×10^{-11} N m^2 kg^{-2} 4.3×10^{-3} pc $M_\odot^{-1}$ (km/s)^2	Fundamental Constants
Planck's constant (h)	6.63×10^{-34} J s	
Boltzmann constant (k_B)	1.4×10^{-23} J K^{-1}	
Elementary charge (e)	1.6×10^{-19} C	
Hydrogen ionization energy (E_0)	13.6 eV	
Stefan-Boltzmann constant (σ)	5.67×10^{-8} W m^{-2} K^{-4}	
Wien constant ($\lambda_{max}T$)	2.898×10^{-3} m K	
Speed of light in vacuum (c)	3×10^{8} ms^{-1}	
Proton mass (m_p)	1.00728 u	
Helium mass (m_{He})	4.0015 u	
Atomic mass unit (u)	1.66×10^{-27} kg	
Astronomical Unit (au)	1.5×10^{11} m	Distances
Parsec (pc)	3.086×10^{16} m	

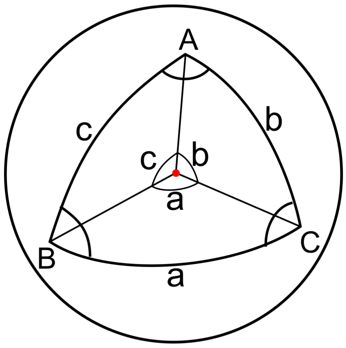
Relevant Formulae

Polar Form of an Ellipse: With the origin centred at the focus, the locus of an ellipse is given by

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

where a is the semi-major axis, e the eccentricity, and θ the angle from the point nearest to the origin.

Spherical Trigonometry: For a spherical triangle $\triangle ABC$ with sides of central angles a, b, and c, we have sine and cosine rules:



$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}\;; \qquad \cos a = \cos b \cos c + \sin b \sin c \cos A\;.$$

Expansions: For small x ($x \approx 0.1 \ll 1$), the following functions have approximations

$$e^x \approx 1 + x + \frac{x^2}{2}\;; \qquad \log(1+x) \approx \frac{x}{\ln 10}{}'\;; \qquad (1+x)^n \approx 1 + nx\;; \qquad \sin x \approx \tan x \approx x\;.$$

Differentiation Rules:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{dg}{dx} + \frac{df}{dx}g(x)\;; \qquad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{df}{dx} - \frac{dg}{dx}f(x)}{g(x)^2}\;; \qquad \frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}\;.$$

Useful Derivatives:

$$\frac{d(x^n)}{dx} = nx^{n-1}\;; \qquad \frac{d(e^x)}{dx} = e^x\;; \qquad \frac{d(\log_a x)}{dx} = \frac{1}{x\ln a}\;;$$

$$\frac{d(\sin x)}{dx} = \cos x\;; \qquad \frac{d(\cos x)}{dx} = -\sin x\;; \qquad \frac{d(\tan x)}{dx} = \sec^2 x\;;$$

$$\frac{d(\csc x)}{dx} = -\csc x \cot x\;; \qquad \frac{d(\sec x)}{dx} = \sec x \tan x\;; \qquad \frac{d(\cot x)}{dx} = -\csc^2 x\;.$$

1. **True or False** [**10 marks**]

Determine if each of the following statements about exoplanetary detection is **True** or **False**. No justifications are necessary for this question.

(a) Exoplanets with a face-on orbit cannot be detected by radial velocity methods, and those with an edge-on orbit cannot be detected by astrometric methods.

(b) Two planets of equal mass orbiting the same star at different distances can produce radial velocity signals of the same amplitude.

(c) If the density of the host star is known, it is possible to constrain the orbital distance of an exoplanet with only a single transit observation.

(d) A detection of a non-zero difference between the light curves of a transit measured at different wavelengths implies a detection of an exoplanetary atmosphere.

(e) Transit timing variations of the same planet can remain negative and become increasingly negative over many successive transits, even if the planet's orbit is gravitationally bound and not decaying.

2. **Star-Chaser** [**13 marks**]

An astronomy enthusiast realises there is a "star" in the sky that does not change its position with respect to the background stars throughout the year and decides to chase after it, constantly adjusting his direction to move directly toward the instantaneous azimuth of the "star".

You, a MOAA finalist, know better – the "star" is in fact a **geostationary satellite**. Nonetheless, it is still interesting to contemplate the crazy path the enthusiastic star-chaser will take as he proceeds on his chase.

You may assume that the Earth is a **perfect sphere**, the "star" is sufficiently **bright to be seen in daylight**, and the enthusiast is unstoppable and will do everything he can to move towards the "star", even if he needs to walk on water. For simplicity, you may neglect near-Earth effects and consider the "star" to be infinitely far away.

(a) Determine the instantaneous horizontal coordinates (h, A) of the "star" when the star-chaser is at coordinate (ϕ, λ) along his path. Express your answer in terms of the satellite's longitude Λ, ϕ and λ.

[5 marks]

(b) Hence, or otherwise, show that the path taken by the star-chaser is a parametric curve

$$\cos(\phi)\sec(\Lambda - \lambda) = K \tag{1}$$

where K is a constant.

[8 marks]

3. **Bon Voyage [15 marks]**

A yacht piloted by Captain Holden left Pelabuhan Penagi, Riau ($3°54'$ N, $108°24'$ E) to Miri Marina, Sarawak ($4°23'$ N, $113°58'$ E) on a direct course. After a day, lightning struck, disabling all electronic navigation and communication. Relying on manual navigation with compass for direction, he kept a heading of $\beta = 84°$ (measured from North to East) at constant speed. He observed Pleiades (M45) crossing the meridian at 22:21 MYT. He recalled it was also at meridian when he left port at 22:51 MYT less than 2 days ago.

Using his knowledge in astronomy and navigation, he estimates that he will arrive at his destination in about 14 hours from his current position.

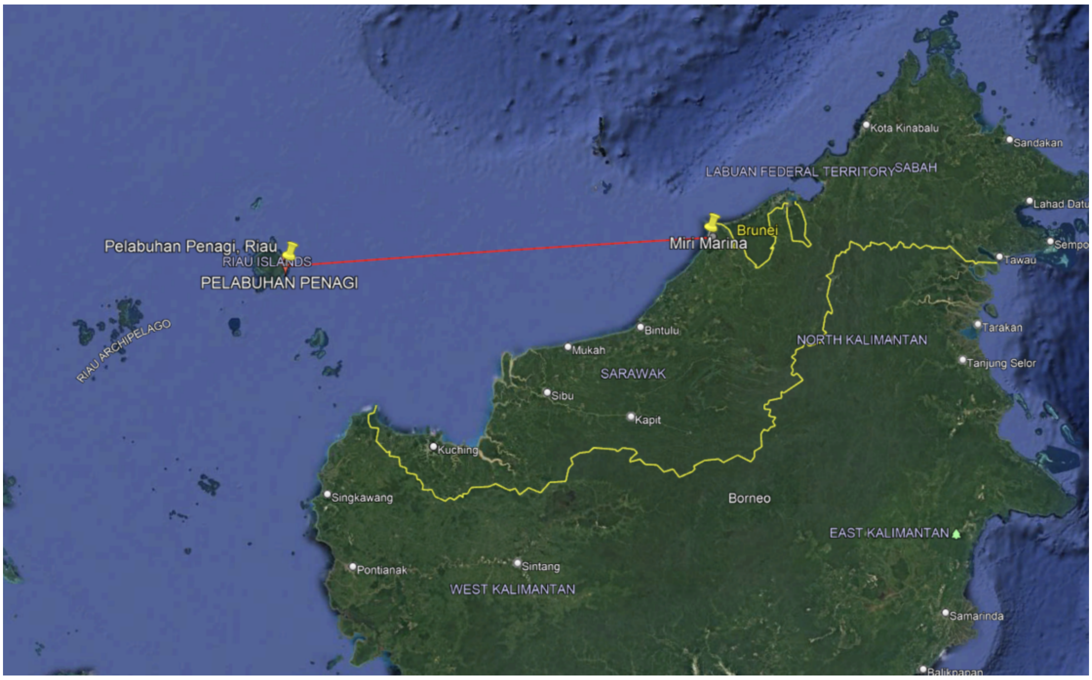


Figure 1: Google Map depiction of Captain Holden's course

(a) The latitude of the yacht's position is $4°25'$ N. Calculate the longitude of the yacht's position rounded to the nearest degree and minute.

[5 marks]

(b) Express the distance along the course L in terms of latitude between two points ϕ_1, ϕ_2, the heading β, and the Earth's radius $R_\oplus$.

[6 marks]

(c) Given the latitude of the yacht's current position is $4°25'$ N and the Earth's radius at this position is 6,378 km, determine if Captain Holden can arrive at Miri in less than an hour. Give a quantitative explanation for your answer, **no marks will be awarded without a calculation.**

[4 marks]

4. **T Coronae Borealis [12 marks]**

T Coronae Borealis (T CrB) is a recurrent nova that hosts a red giant (RG) and a white dwarf (WD) in a non-eclipsing binary system. During its quiescent phase, its V-band magnitude varies sinusoidally between 9.4 - 10.8, due to apparent changes in the surface area of the tidally distorted RG. A V-band magnitude of 12.3 has been observed 1.1 ± 0.3 year before its previous eruption in 1946. This dimming is now known as the pre-eruption dip. A similar pre-eruption dip was observed in early May, 2023.

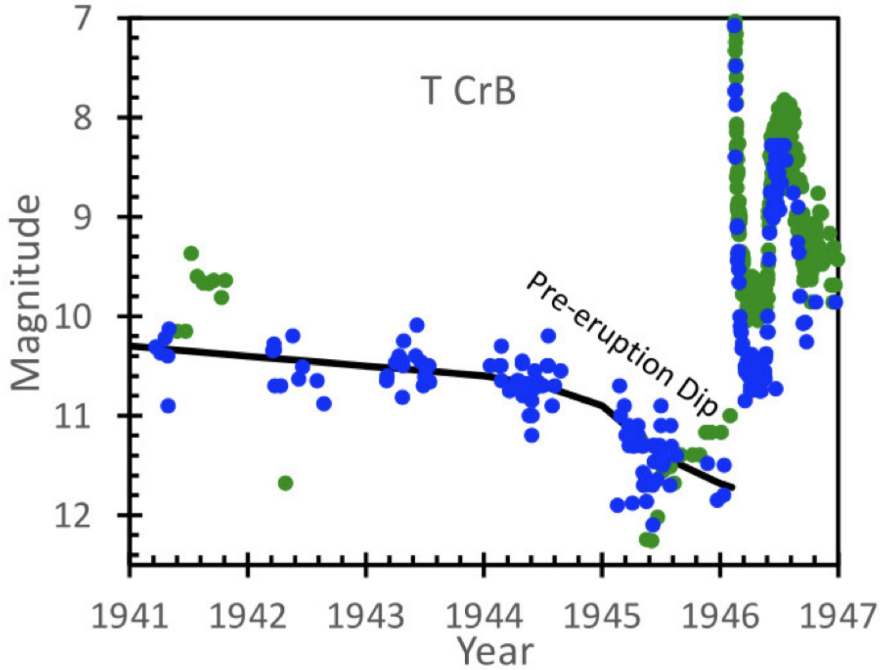


Figure 2: Light curve of T CrB during the 1946 eruption

Since there is no known mechanism by which the RG can suddenly fade by such a significant amount, the dimming is most likely extrinsic to the RG. During the peak of its eruption on February 9, 1946, T CrB reached a peak V-band magnitude of 3.0. Properties of T CrB are listed in the table below.

RG effective radius ($R_\odot$)	RG effective temperature (K)	Parallax (milliarcsec)
63.5 ± 0.3	3561 ± 3	1.0920 ± 0.0275

(a) Calculate the mean absolute magnitude of the binary system during the quiescent phase.

[4 marks]

(b) The luminosity of the RG is unchanged during the 1946 eruption. Calculate the V-band luminosity of the WD during the eruption.

[6 marks]

(c) If the behaviour of T CrB did not change since the 1946 eruption, provide an estimate of the time of the next eruption. (Give your answer in terms of the calendar year, with precision up to one decimal point.)

[2 marks]

5. **In Space, No One Can Hear You Scream [40 marks]**

In space, no one can hear you scream! So in the unfortunate event that you fall out of the International Space Station (ISS) without a tether, make sure you bring enough oxygen!

A clumsy astronaut did not heed this advice and fell off the ISS at a velocity of $v_0 = 0.2$ m/s towards the Earth. You may assume that the ISS revolves around the Earth in a circular orbit at a height negligible in comparison to the Earth's radius, and his fall did not impart any significant change to the orbit of the ISS.

(a) Calculate the velocity and the orbital period of the ISS.

[4 marks]

(b) Determine the minimum duration at which the astronaut's oxygen tank should last to get him safely back to the ISS.

[5 marks]

(c) Determine the eccentricity of the astronaut's orbit.

[3 marks]

(d) At what point was the astronaut furthest away from the ISS? Was the astronaut ahead or behind the ISS at this point?

[12 marks]

(e) Once the astronaut was rescued and interrogated, he denied that it was an accident and said he just wanted to head home to Earth. The rest of the ISS crew were furious and threw him back towards Earth. (Don't do this!)

For the rest of the question, take the height of ISS' orbit to be $H = 400$ km.

i. Calculate the minimum velocity $V_{\min}$ for the thrown-out astronaut to land on Earth.

[10 marks]

ii. The crew threw the astronaut out at a velocity $V > V_{\min}$. Determine the minimum time $T_{\min}$ it could take for the astronaut to land on Earth.

[6 marks]

6. **Cosmology with the Lyman α Forest [50 marks]**

New measurements from the Dark Energy Spectroscopic Instrument (DESI) made headlines this year, claiming evidence for dark energy to vary across cosmic time, challenging the traditional assumptions of a cosmological constant. To obtain this result, cosmologists have to measure the Hubble parameter at different redshifts, i.e.

$$H(z) = H_0 \left[\sum_i \Omega_i(z) \right]^{1/2} , \tag{2}$$

where the subscript i denotes the different components of the universe, and $\Omega_i(z) = \rho_i(z)/\rho_c$ is the ratio between the energy density of the component and the critical density of the universe $\rho_c = 3H_0^2/8\pi G$. The subscripts "0" denote quantities measured today.

(a) Assume that the energy density of the universe today is composed of 70% dark energy and 30% matter.

i. The transition between the matter-dominated epoch to the dark energy-dominated epoch occurs when the energy densities of the two components are equal. Calculate the redshift $z_{\rm eq}^{\rm CC}$ at which this occurs, assuming dark energy is a cosmological constant.

[3 marks]

ii. The most conservative analysis of DESI data prefers an evolving dark energy, where the energy density evolves with redshift as

$$\Omega_{\rm DE}(z) = \Omega_{\rm DE,0}(1+z)^{3(1+w_0+w_a)} \exp\left(-\frac{3w_a z}{1+z}\right) , \tag{3}$$

where $w_0 = -0.4$ and $w_a = -1.7$. Determine if the transition redshift $z_{\rm eq}^{\rm Evol}$ for evolving dark energy is **larger than / smaller than / the same as** $z_{\rm eq}^{\rm CC}$.

[2 marks]

(b) At different redshifts, different objects (tracers) were used to measure $H(z)$. At the largest redshifts, these were done through observations of the Lyman α (Lyα) forest, a sequence of absorption lines found in spectra of distant quasars, created when hydrogen atoms along the line of sight are excited by the quasar's light. An example spectra is shown in Fig. 3

i. The Lyα line is associated to electronic transitions of hydrogen atoms between the ground state and the first excited state. Calculate the rest frame wavelength of the Lyα line.

[4 marks]

ii. In a non-expanding universe, a quasar has a specific intensity $I_0(\lambda)$. A uniform cloud of hydrogen gas resides between the Earth and the quasar. The proper length of the cloud is $\Delta L = 10$ pc along the line of sight, the Lyα absorption has a cross-section $\sigma = 10^{-2}$ cm^2, and the number density of hydrogen atoms in the ground state is $n_{\rm HI} = 3 \times 10^{-6}$ cm^{-3}. The observed specific intensity of the quasar has a form

$$I(\lambda) = I_0(\lambda)F(\lambda) , \tag{4}$$

where $F(\lambda)$ is the transmission. Sketch a labelled plot of $F(\lambda)$. You may neglect the width of the Lyα line.

[5 marks]

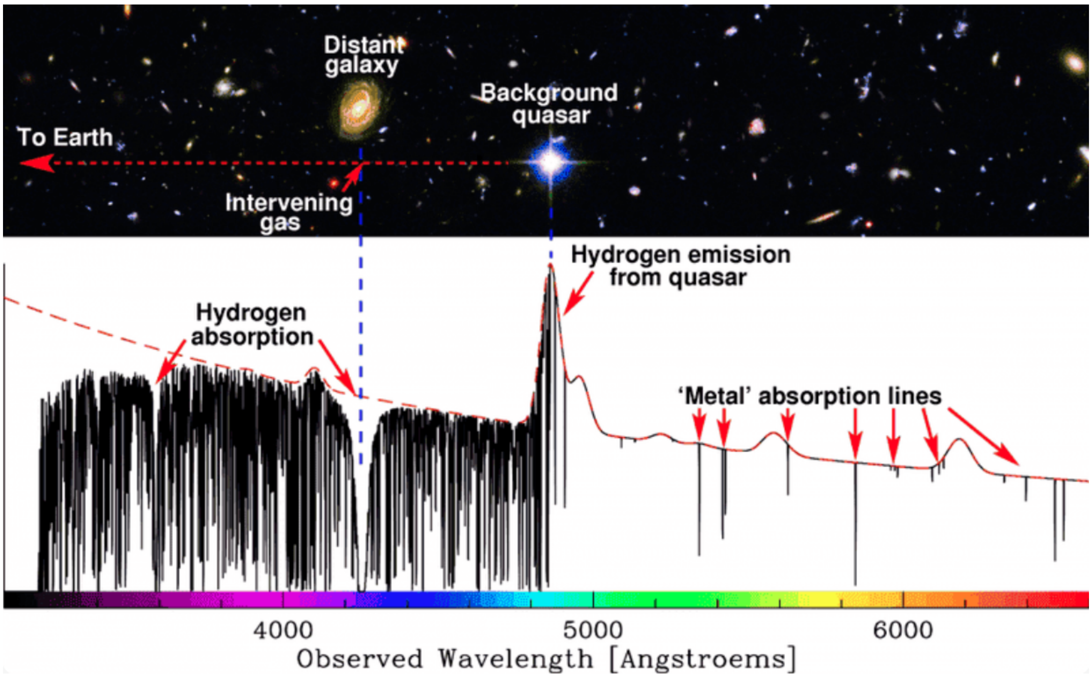


Figure 3: Lyman α forest and its formation

iii. In reality, the universe has been expanding in time at a rate determined by $H(z)$. Throughout most of the formation of the Lyα forest, the universe is matter dominated.

α) Show that the proper length ΔL at which the light from the quasar travels through the cloud of hydrogen gas varies at redshift as

$$\Delta L(z) = \frac{c\Delta z}{H_0 \Omega_{m,0}^{1/2} (1+z)^{\beta}} , \tag{5}$$

where $\Omega_{m,0}$ is the present day matter density parameter, Δz is the effective redshift interval (width) of the Lyα absorption line by the cloud, and β is a rational number that you should determine.

[7 marks]

β) Suppose the transmission of a quasar features an absorption line $F(\lambda = 5000\text{Å}) = 0.3$, calculate the redshift z_0 at which this absorption line was formed, and the number density of ground state hydrogen atoms $n_{\mathrm{HI}}(z_0)$ at this redshift along the line of sight.

Hint: Your transmission should scale as $F(\lambda) = \exp\big(-A\lambda^{3/2}\big)$.

[12 marks]

γ) Knowing that baryons make up only 15% of matter in the universe, of which 25% of the mass is comprised of helium, calculate the fraction of hydrogen atoms that were in the ground state at redshift z_0.

[8 marks]

(c) As shown in Fig. 3, the absorption spectrum due to the Lyα forest is not uniform, but instead shows fluctuations caused by inhomogeneities in the intervening gas $n_{\rm HI}$. The two-point correlation function (2PCF) $\xi(\vec{s})$ between these fluctuations, i.e. the excess probability (compared to random) that two fluctuations in $n_{\rm HI}$ are separated by a comoving distance $\vec{s}$, is measured by DESI and is shown in Fig. 4.

The 2PCF features a distinct peak, signalling an increased tendency for hydrogen clouds to cluster at that separation. This is caused by baryonic acoustic oscillations (BAO) in the early universe, and is a universal scale for clustering of all types of celestial objects. This allows it to be used as a "standard ruler" of length $s_{\rm BAO}$.

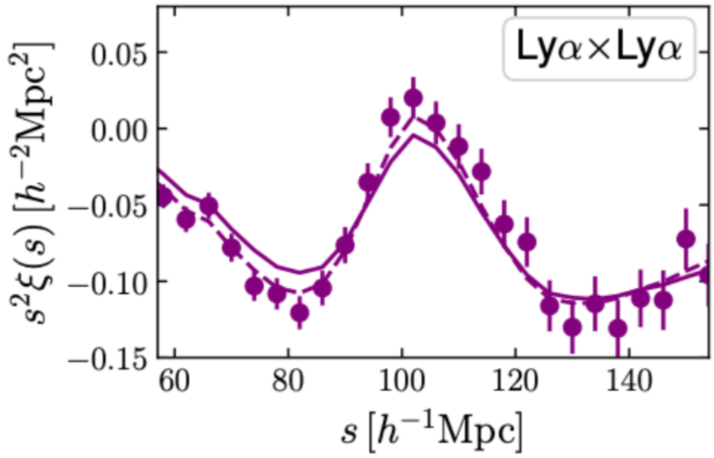


Figure 4: DESI measurement of the Lyα forest 2PCF in unis of comoving distance. The 2PCF is rescaled by s^2 to better contrast the BAO peak. Here the units of comoving distance is given by Mpc/h, defined through $H_0 = 100h$ km/s/Mpc.

i. What is the comoving size of BAO (in units of Mpc/h) as measured by DESI in the Lyα forest? Provide an uncertainty estimate with your answer.

[2 marks]

ii. The BAO scale can be measured in the 2PCF both along the line of sight across redshifts $s_{\rm BAO}^{\parallel}$, and perpendicular to the line of sight $s_{\rm BAO}^{\perp}$ through the angle it subtends. If we were to take a snapshot of the current universe, these two should be equal, i.e. $s_{\rm BAO}^{\parallel} = s_{\rm BAO}^{\perp}$. However, we infer distances through redshifts, which in addition to Hubble expansion, gains contribution from the peculiar velocity of the object.

α) Suppose an object at comoving distance r has a peculiar velocity v. What is the inferred comoving distance of the object through its redshift?

[6 marks]

β) The effect of peculiar velocities on the inferred distances are named redshift space distortions (RSD). Peculiar velocities of matter are not random. Instead, they gravitate towards overdense regions. Determine if $\boldsymbol{s}_{\bf BAO}^{\parallel} < \boldsymbol{s}_{\bf BAO}^{\perp}$ / $\boldsymbol{s}_{\bf BAO}^{\parallel} = \boldsymbol{s}_{\bf BAO}^{\perp}$ / $\boldsymbol{s}_{\bf BAO}^{\parallel} > \boldsymbol{s}_{\bf BAO}^{\perp}$ due to RSD effects.

[1 mark]