malaysia olympiad on astronomy and astrophysics

Theoretical Round

Good Luck and Clear Skies!

General Instructions

1. The **Theoretical Round** consists of 6 questions on astronomical theory.
   The allocated time for this round is **180 minutes**.
2. The round will be announced to start by the invigilator (Camp Facilitator).
   Please **DO NOT** turn over this page before the start of this round.
3. A countdown timer would be shown on the screen. There are no restrictions to the time budget for each question.
4. You will be provided working sheets for your solutions. On **EACH** working sheet, write down your **Name**, **IC Number** and **Question Number**. All answers that are to be evaluated must be written on the sheets provided. You are advised to use separated pages for separate questions.
5. **Cross out** sections that you do not want to be evaluated.
6. Use as many mathematical expressions to help the graders better understand your solutions. The graders may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
7. You are not allowed to leave your working desk without permission. If you need any assistance (malfunctioning calculator, restroom visits, insufficient or missing sheets, etc.), please put up your hand to signal the invigilator.
8. The round would end once the countdown timer rings. At the end of the round, you must stop writing immediately. Sort and put your working sheets and graph paper in one stack. Put papers you do not want to be graded in another stack. You are allowed to keep this question paper.
9. Please remain seated until your papers are collected. You are allowed to leave once all papers are collected.
10. The mark distribution is as follows:

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<tr><th>Question</th><th>Subpart</th><th>Marks</th><th>Question</th><th colspan="2">Subpart</th><th>Marks</th><th>Question</th><th colspan="2">Subpart</th><th>Marks</th></tr>
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Table of Constants

Mass ($M_\oplus$) Radius ($R_\oplus$) Obliquity (ε)	6×10^{24} kg 6.4×10^{6} m $23.5°$	Earth
Mass ($M_\odot$) Radius ($R_\odot$) Luminosity ($L_\odot$) Surface Temperature ($T_\odot$) Apparent V-band magnitude ($m_{\odot V}$) Absolute V-band magnitude ($\mathcal{M}_\odot$)	2×10^{30} kg 6.96×10^{8} m 3.82×10^{26} J 5778 K -26.8 mag 4.83 mag	Sun
Gravitational constant (G) Planck's constant (h) Electrostatic constant (k_e) Elementary Charge (e) Hydrogen Ionization Energy (E_0) Stefan-Boltzmann constant (σ) Wien constant ($\lambda_{\max} T$) Hubble constant (H_0) Speed of light in vacuum (c) Proton Mass (m_p) Electron Mass (m_e)	6.67×10^{-11} N m^2 kg^{-2} 4.3×10^{-3} pc $M_\odot^{-1}$ (km/s)^2 6.63×10^{-34} J s 9×10^{9} N C^{-2} 1.6×10^{19} C 13.6 eV 5.67×10^{-8} W m^{-2} K^{-4} 2.898×10^{-3} m K 70 (km/s) Mpc^{-1} 3×10^{8} ms^{-1} 1.7×10^{-27} kg 9.1×10^{-31} kg	Fundamental Constants
Astronomical Unit (au) Parsec (pc)	1.5×10^{11} m 3.086×10^{16} m	Distances

Relevant Formulae

Spherical Trigonometry: A spherical triangle ΔABC are triangles on the surface of a sphere, which sides a, b, c are arc segments of great circles. They obey trigonometric relations:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{1}$$

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{2}$$

Magnitudes: The apparent magnitude of an object, m is related to the ratio of its flux F to a reference flux F_0, via the Pogson's relation:

$$m = -2.5 \log\left(\frac{F}{F_0}\right) \tag{3}$$

The absolute magnitude, $\mathcal{M}$, of an object is defined to be equal to its apparent magnitude when viewed from 10 pc away, which yields the distance modulus, μ:

$$\mu = m - \mathcal{M} = 5\log(d/pc) - 5 \tag{4}$$

Virial Theorem: The virial theorem states that for a stationary system of discrete particles bound by gravitational forces, has a time-averaged total kinetic energy $\langle K \rangle$ obeys

$$\langle K \rangle = -\frac{1}{2}\langle G \rangle \tag{5}$$

where $\langle G \rangle$ is the time-averaged gravitational potential energy of the system.

Stefan-Boltzmann Law: The power radiated by a blackbody of temperature T and surface area A is given by

$$P = A\sigma T^4 \tag{6}$$

1. **(Guesstimation!)**

Give order-of-magnitude estimates for the problems below. Marks would be given for reasonable modelling.

(a) **(Electric Spheres)**

You would often see in astronomy, that we assume that stars and planets are electrically neutral, i.e. they carry no net charge. Convince this yourself by coming up with an estimate for the net charge limit of the Sun.

(b) **(Sgr A*)**

By now, you should have seen the latest image of the supermassive black hole in the center of our galaxy, Sgr A*. The image is taken by the Event Horizon Telescope Collaboration (EHT) using a Earth-size telescope. In comparison with the image of M87* released in 2019, Sgr A* exhibits a higher variability, making it harder to image. Given that the mass of Sgr A* is about $4 \times 10^7 M_\odot$, estimate the typical timescale at which Sgr A* varies its structure.

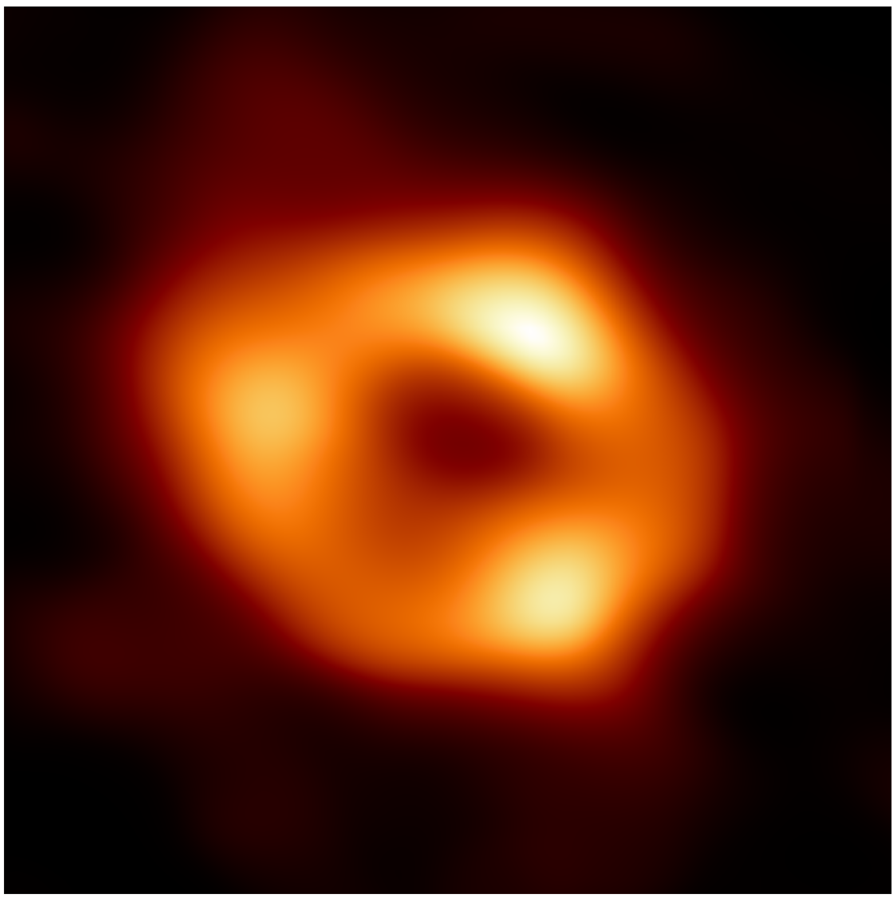

Figure 1: Sgr A* (Credit: EHT Collaboration)

(c) **(Gamma-Ray Bursts)**

Gamma-ray bursts (GRBs) are energetic explosions that emit large amount of gamma radiation. At low redshifts, they are observed at a rate of 0.44 Gpc^{-3} yr^{-1}. The spatial density of galaxies is about 3×10^{-3} Mpc^{-3}, and our galaxy can be approximated as a disc, 30 kpc in diameter. Assume that a GRB within 2 kpc of Earth would be dangerous to life on Earth. Estimate the number of dangerous GRBs that Earth has experienced in its lifetime.

2. **(Aberrations from a Narrow-Band Filter)**

During your visit to Star-Finder's IOP (Imaging and Observation Party), you have mounted a CCD camera onto a focused Cassegrain telescope and prepared for imaging. However, as you direct your scope to a planetary nebula, Fongky advised you to take images with narrow-band filters. The filters are coloured glass plates (refractive index $n = 1.5$), inserted in front of the CCD, and are typically $t = 1$ mm thick.

(a) In the paraxial approximation, where we only consider small angular deviations from the optical axis, by how much should you have to adjust the focus of your telescope after the insertion of a narrow-band filter?

(b) You should have no problem doing this, as your telescope is small. However, large telescopes may have these heavy instruments fixed on the telescope. Instead, they adjust the focal length by moving the secondary mirror. By how much should the secondary mirror be moved to regain focus?

(c) Focal shifts are not the only problem with the insertion of a narrow-band filter. Far from the paraxial approximation, light rays enters the band at different incident angles i, resulting in aberrations. Derive the focal shift in question (a) as a function of incident angle i.

3. **(J0313-1806)**

J0313-1806 is the current record holder for the highest redshift quasar discovered so far. Its spectrum is shown below.

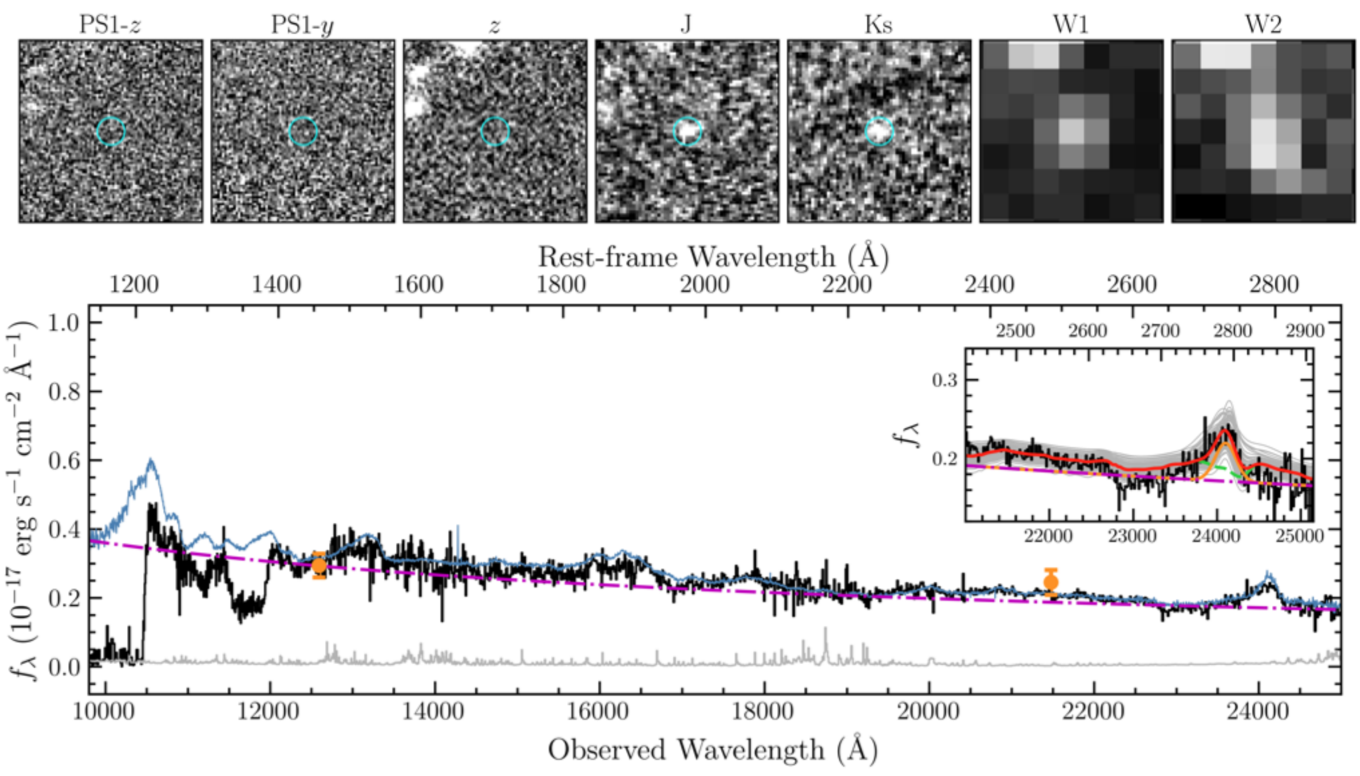

Figure 2: Top panel: Image cutouts for J0313–1806 in PS1 z, PS1 y, DELS z, VISTA J, VISTA Ks, WISE W1 and WISE W2 bands. Bottom panel: Stacked spectrum of J0313–1806. The blue line denotes the quasar composite spectrum. The purple dashed line denotes the power-law continuum. The orange points are flux densities determined from photometry in the J and Ks-bands. The inset panel shows the Mg II line fitting with the purple dot-dashed line denoting the power-law continuum, the green dashed line denoting the pseudo-continuum model (the sum of power law continuum, Fe II emission, and Balmer continuum), the orange line representing the Gaussian fitting of the Mg II line and the red line representing the total fit of pseudo-continuum and Mg II line. (Wang et al. 2021)

(a) In the top panel of Figure 2, J0313-1806 is not detected in the z and y-bands but appear bright in other bands. Using this information (or otherwise), determine the redshift of J0313-1806. (**Hint: The Ly-α transition is an (de-)excitation of the hydrogen atom between the $n = 1$ and $n = 2$ electronic states.**)

(b) At a given redshift z, the Hubble parameter depends on the energy density of the universe by the Friedman equation

$$H^2(z) = \frac{8\pi G}{3}\rho(z) \tag{7}$$

At such high redshift as J0313-1806, the universe can be approximated to be matter-dominated. Determine the age of the universe at the redshift of J0313-1806. Express your answer in Myr.

(c) The formation of supermassive black holes, like Sgr A* and the central black hole in quasars, are still unclear. Astronomers postulate that their mass grows exponentially as $M_{\rm BH} = M_{\rm seed} e^{t/\tau}$, where the Salpeter timescale τ is the characteristic timescale for a black hole seed to e-fold its mass,

$$\tau = 45\left(\frac{\eta/(1+\eta)}{0.1}\right)\left(\frac{L}{L_{\rm Edd}}\right)^{-1} \text{Myr} \tag{8}$$

where η is the radiative efficiency and L_{Edd} is the Eddington luminosity. The central BH of J0313-1806 is estimated to be around $1.6 \times 10^9 M_\odot$. How long does it take for a massive seed of $100 M_\odot$ to grow to this mass? Is there enough time for the universe to grow a black hole of this mass?

(d) In an attempt to beat this record holder, you have handful of high-redshift quasar candidates you would like to confirm with spectroscopy with the 10-m Keck telescope at Mauna Kea (19.8°N, 155°W) in mid-June. The RA and DEC of the candidates are as follows:

i. 02 h 52 m, 21° 30'

ii. 05 h 38 m, -15° 27'

iii. 15 h 42 m, 09° 28'

iv. 19 h 30 m, 30° 30'

v. 20 h 20 m, -75° 15'

Given that you know a star with RA 17h will cross the meridian around midnight at Keck, which of the above is(are) viable target(s)?

4. **(James Webb Space Telescope)**

On December 2021, the James Webb Space Telescope (JWST) was launched to the second Lagrange point (L2) as a successor to the Hubble Space Telescope. Upon its approach to L2 a month later, a picture of JWST was taken by the astronomer Albert Kong from NTHU. The images and the coordinates of the telescopes used are shown in Figure 3.

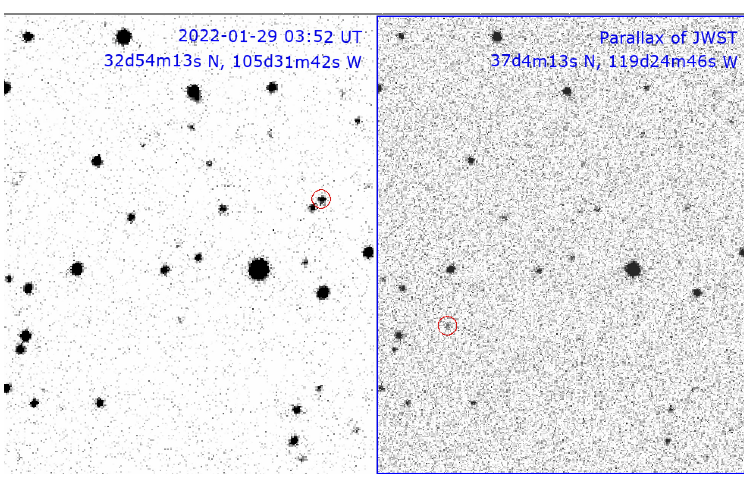

Figure 3: Images of JWST (circled) taken using telescopes at coordinates at the top right corner of the respective images.

(a) Determine the straight line distance between the two observing sites.

The Lagrange points are 5 points in space where an object with mass much smaller than the Earth and the Sun ($m \ll M_\odot, M_\oplus$) appears to be fixed in the sky. L2 marks the point away from both the Earth and the Sun, as depicted in Figure 4.

(b) Determine the distance of L2 from Earth. You may find the binomial approximation, $(1+x)^n \approx 1 + nx$ for $x \ll 1$ helpful.

(c) With the distance you obtained in (b), determine the angular separation, in arcsec, for JWST in these two images.

(d) In reality, the JWST does not appear stationary in Earth's co-rotating frame. Instead, it circles around L2 in a large enough orbit to avoid Earth's shadow. What is the minimum radius r should this orbit be, to avoid Earth's shadow at all times?

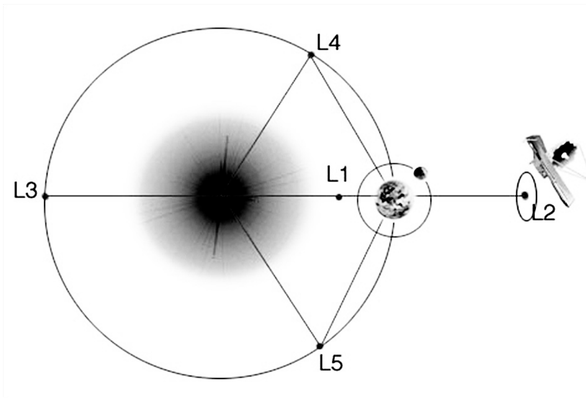

Figure 4: The Lagrange points with the JWST at L2

5. **(Summers and Winters)**

It is a common misconception that the temperature differences in summers and winters is due to the elliptic orbit of Earth. Instead we would show in this question that the seasons is due to the obliquity of the Earth. You may safely assume that the Earth is a rapidly rotating spherical body, constantly in thermodynamic equilibrium, with an albedo $A = 0.31$.

(a) Determine the ratios between the equilibrium temperature of the Earth at perihelion and aphelion. The eccentricity e of Earth's orbit is 0.0167.

(b) In part (a), you have treated the temperature to be uniform across the Earth. Instead, each circle of constant latitude can be treated as being in local thermodynamic equilibrium. Consider 'ribbons' of latitude with thickness $\Delta\phi$. Find the temperature of the Earth as a function of latitude $T(\phi)$ when the Earth is at vernal equinox.

(c) If an alien civilization wants to observe Earth via direct detection, in what wavelength should they observe in?

(d) During the summer solstice, the Sun is normally incident on the Tropic of Cancer. In which latitudes does the Sun never set? Derive the latitude-dependent temperature $T(\phi)$ for these latitudes.
(**Hint**: For small angles $\theta \ll 1$, you may approximate $\sin\theta \approx \theta$, $\cos\theta \approx 1$)

(e) This model for Earth's seasonal variation of temperature is crude and non-physical. From your calculations above, **explain** why this model gives non-physical results and **propose** an improvement to the model.

6. **(Gauging Dark Matter in A Globular Cluster)**

Astronomers now believe (with high level of confidence) that the Milky Way, like all galaxies, is surrounded by a dark matter halo, a cosmological structure made up of particles that only interact via gravity. Inside of galaxies, we find globular clusters, dense collections of stars also bound together by gravity. Astronomers believe globular clusters may also be the homes of smaller dark matter subhalos, which is usually gauged by the mass-to-light ratio of the cluster. In this question, you will go through the problems that will pop up in such an observational analysis.

(a) **(Planning an Observation)**

You are planning to observe the MOAA-22 globular cluster on the night of the IOAA exam (15 August 2022), at Kutaisi, Georgia (latitude $\phi = 42°$N). The equatorial coordinates of the cluster is $(\alpha, \delta) = (22^{\text{h}}, 54°30')$.

i. Approximately, in what constellation is MOAA-22 located?

ii. For good seeing, we prefer to perform observations on targets at least 30° above the horizon, and at least 1 hour before/after sunrise/sunset. The operator has kindly told you that the sunset/sunrise times would be 8 p.m. and 6 a.m. respectively, and the Local Sidereal Time (LST) during midnight would be $20^{\text{h}}30^{\text{m}}$. Determine the maximum duration of observation available to you. (**Hint: You may want to recall coordinate conversions from equatorial to horizontal coordinates.**)

iii. On the Color-Magnitude Diagram (CMD) of the cluster, you notice a massive star of spectral class A5 on the main-sequence. A5 stars have a typical absolute V-band magnitude of $M_V = 2.2$ and intrinsic colour $(B - V)_0 = 0.14$. The observed magnitude is $V = 10.5$ with a colour index $B - V = 0.25$. Estimate the distance of MOAA-22, assuming a total-to-selective extinction ratio $R_V = A_V/E_{BV} = 3.2$, where E_{BV} is the colour excess.
If you cannot get an answer for this part, use $d = 0.4$ kpc for the rest of the question.

iv. The conventions of galactic longitude and latitude (ℓ, b) is shown in Figure 5. Note that the galactic plane rotates clockwise in the given figure. The galactic coordinates of MOAA-22 is $(99.67°, -0.43°)$. Calculate the line-of-sight velocity of MOAA-22, assuming it moves with the general galactic rotation. By how much is MOAA-22 redshifted or blueshifted from the galactic motion? (**Hint: The Galactic rotation curve remains flat at 220 km/s beyond 5 kpc from the Galactic Centre.**)

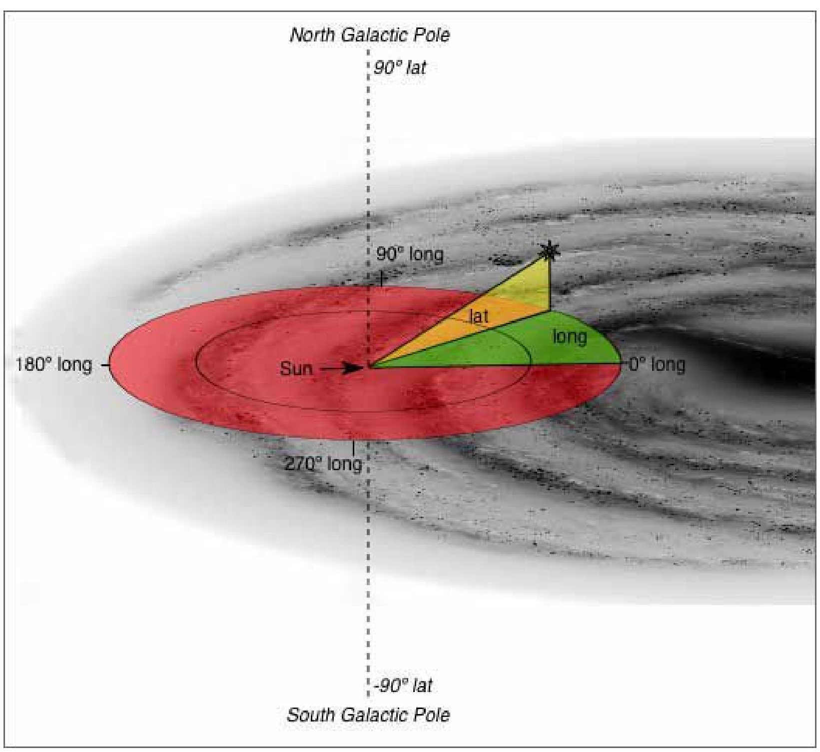

Figure 5: Conventions of Galactic Coordinates

v. The conventions of ecliptic longitude and latitude (λ, β) is shown in Figure 6. The conversion formulae from equatorial to ecliptic coordinates are:

$$\sin\beta = \sin\delta\cos\varepsilon - \cos\delta\sin\varepsilon\sin\alpha\ ; \qquad \cos\lambda = \frac{\sin\delta - \cos\varepsilon\sin\beta}{\sin\varepsilon\cos\beta}, \tag{9}$$

where ε is the obliquity of Earth. Find the ecliptic coordinates of MOAA-22.

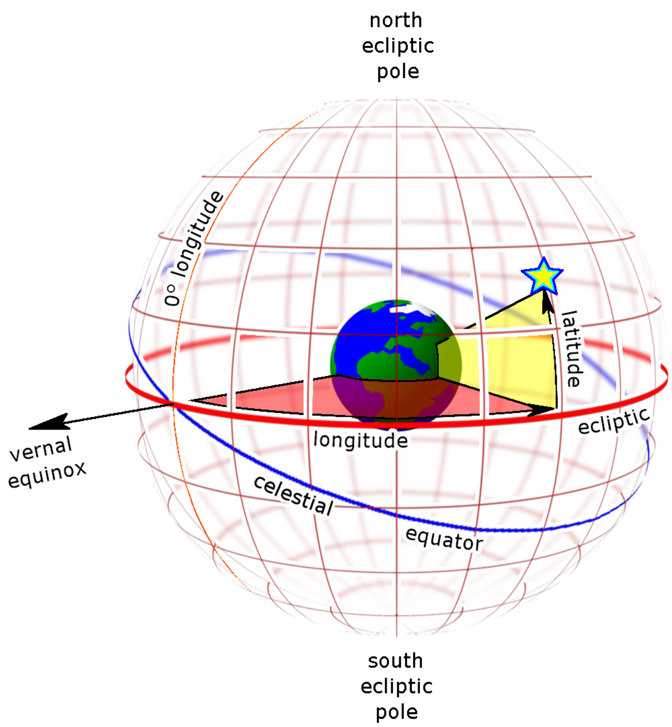

Figure 6: Conventions of Ecliptic Coordinates

vi. Make a sketch of Earth's orbit as seen from the North Ecliptic Pole. Clearly indicate the **vernal equinox**, **Earth's position during the observation**, and **the position of MOAA-22**. In ecliptic coordinates, what direction $(\lambda_\oplus, \beta_\oplus)$ is the Earth moving during the observation? Draw an arrow indicating the direction on your sketch.

vii. The Earth's orbital velocity is about 30 km/s. What is the additional redshift/blueshift of the cluster due to the Earth's motion around the Sun?

viii. In addition, the Sun has a peculiar motion relative to the local standard of rest, which adds onto its mean motion around the Galactic Centre. The direction of this motion, known as the solar apex, points at equatorial coordinates $(18^{\mathrm{h}}30^{\mathrm{m}}, 30°\mathrm{N})$ near Hercules. What is the additional redshift/blueshift of the cluster due to the solar apex?

ix. Compare your answers in (iv), (vii) and (viii). Are these calculations relevant, if we want to calculate the peculiar velocity of MOAA-22? What about the redshifts/blueshifts due to the rotation of the Earth? Give a brief explanation.

(b) **(Dynamics of A Globular Cluster)**

After your data reduction in part (a), you found out that MOAA-22 has residual redshift due to peculiar motion. You went on to perform the subsequent astrophysical measurements to gauge the dark matter content in MOAA-22, with help from your colleague, Dr. Tong Ko-Song.

i. From spectroscopic measurements of 12 cluster members, you obtained from Doppler shift measurements, the line-of-sight velocities of the stars (in km/s) as tabulated below. Determine the residual redshift, and the velocity dispersion (variance of velocity distribution) of MOAA-22.

-1.0	17.3	-16.3	6.5
-1.7	13.9	13.3	-13.4
2.2	14.0	0.2	10.0

Table 1: Line-of-sight Velocities of 12 Cluster Members

ii. Dr. Tong Ko-Song thinks that your velocity dispersion calculations are wrong, because your dataset includes residual redshift. Do you agree with Dr. Tong? Briefly explain.

iii. Angular measurements yield the mean radial angular distance of stars is about 1° from the center of the cluster, which we will call the radius of MOAA-22. Calculate the radius of MOAA-22 in units of parsec. **(Hint: If you did not solve (aii), let the distance to MOAA-22 be 0.4 kpc.)**

iv. Determine the mean density of MOAA-22 in units of $M_\odot/\text{parsec}^3$.

v. You measure the surface brightness of MOAA-22 to be 10.28 mag/arcsec^2. Determine the luminosity density of the cluster in units of $L_\odot/\text{parsec}^2$.

vi. Determine the mass-to-light ratio of MOAA-22 in units of $M_\odot/L_\odot$. Pay attention to the units involved.

vii. Dr. Tong Ko-Song believes that you have found dark matter in MOAA-22 and proudly submitted your findings to Nature Astronomy. He asked you if you agree to his conclusion. What are your comments?