



Theoretical Round

Good Luck and Clear Skies!

General Instructions

1. The **Theoretical Round** consists of 8 questions on astronomical theory.
The allocated time for this round is **180 minutes**.
2. A countdown timer would be shown on the screen. There are no restrictions to the time budget for each task.
3. You need to prepare your own working sheets for your solutions. On **EACH** working sheet, write down your **Name**, **IC Number** and **Question Number**. You are advised to use separated pages for separate questions. All answers that are to be evaluated must be written on these sheets.
4. Use as many mathematical expressions as you think may help the graders to better understand your solutions. The graders may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
5. Enter your final answers in the field provided in the Google Form here: <https://forms.gle/8pgpQd8WHjm31Fzz7>. You may leave the field blank if you do not get a final answer.
6. You are not allowed to leave the camera view without permission. If you need any assistance (restroom visits, connection problems etc.), please put up your hand to signal the invigilator. The invigilator will communicate with you through chat.
7. The round would end once the countdown timer rings. At the end of the round, you must stop writing immediately. You will be given a time limit of 30 minutes to upload all your working sheets in the Google Form above. You are advised to separate the uploaded sheets by question.
8. The mark distribution is as follows:

Question	Subparts	Marks	Question	Subparts	Marks	Question	Subparts	Marks	
1	a	4	5	a	2	8	a	i	3
	b	3		b	6			ii	2
	c	3		c	2			iii	2
2	a	3	6	a	1		b	i	2
	b	2		b	2			ii	4
	c	3		c	3			iii	4
	d	2		d	4			iv	3
3	a	6		7	e			5	c
	b	4	a		2		i	6	
4	a	2	b		1		Total	ii	4
	b	3	c		2			iii	3
	c	1	d		4	115			
	d	4	e		3				
			f	3					

Table of Constants

Mass ($M_{\oplus}$)	6×10^{24} kg	Earth
Radius ($R_{\oplus}$)	6.4×10^6 m	
Mass ($M_{\odot}$)	2×10^{30} kg	Sun
Radius ($R_{\odot}$)	6.96×10^8 m	
Luminosity ($L_{\odot}$)	3.82×10^{26} J	
Surface Temperature ($T_{\odot}$)	5778 K	
Apparent V-band magnitude ($m_{\odot V}$)	−26.8 mag	
Absolute V-band magnitude ($\mathcal{M}_{\odot}$)	4.83 mag	
Gravitational constant (G)	6.67×10^{-11} N m ² kg ^{−2}	Fundamental Constants
Planck's constant (h)	4.3×10^{-34} J s	
Boltzmann constant (k_B)	1.38×10^{-23} J K ^{−1}	
Molar Gas constant (R)	8.31×10^3 J K ^{−1} mol ^{−1}	
Stefan-Boltzmann constant (σ)	5.67×10^{-8} W m ^{−2} K ^{−4}	
Wien constant ($\lambda_{\max} T$)	2.898×10^{-3} m K	
Speed of light in vacuum (c)	3×10^8 ms ^{−1}	
Astronomical Unit (au)	1.5×10^{11} m	Distances
Light-year (ly)	9.46×10^{15} m	
Parsec (pc)	3.086×10^{16} m	

Relevant Formulae

Kepler's Third Law: For an elliptical orbit, the square of the period, T , of an object about the focus is proportional to the cube of the semi-major axis a , such that

$$T^2 = \frac{4\pi^2}{GM} a^3 \quad (1)$$

Magnitudes: The apparent magnitude of an object, m is related to the ratio of its flux F to a reference flux F_0 , via the Pogson's relation:

$$m = -2.5 \log \left(\frac{F}{F_0} \right) \quad (2)$$

The absolute magnitude, $\mathcal{M}$, of an object is defined to be equal to its apparent magnitude when viewed from 10 pc away, which yields the distance modulus, μ :

$$\mu = m - \mathcal{M} = 5 \log(d/\text{pc}) - 5 \quad (3)$$

Virial Theorem: The virial theorem states that for a stationary system of discrete particles bound by gravitational forces, has a time-averaged total kinetic energy

$$\langle K \rangle = -\frac{1}{2} \langle G \rangle \quad (4)$$

where $\langle G \rangle$ is the time-averaged gravitational potential energy of the system.

Blackbody Radiation: The spectrum of a blackbody represents a Planck distribution, given by

$$B_{\lambda}(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \quad \text{or} \quad B_{\nu}(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} \quad (5)$$

where a total power of $F = \sigma T^4$ is emitted per unit surface area, where σ is the Stefan–Boltzmann constant.

Equipartition Theorem: For a gas at temperature T , each degree of freedom (vibrational, translational, rotational) contributes an energy per particle of

$$E_{\text{thermal mode}} = \frac{1}{2} k_B T \quad (6)$$

1. (Guesstimation!)

(a) (A Very Hot Sun)

The Sun, as like all stars are in hydrostatic equilibrium.

Using this fact, give an order of magnitude estimate for the internal temperature of the Sun.

(b) (In A Cluster Far Far Away...)

Figure 1 below shows two color-magnitude diagrams of (a) nearby stars, constructed from measurements by the Hipparcos satellite, and (b) a nearby globular cluster. From the plots below, estimate the distance to the globular cluster.

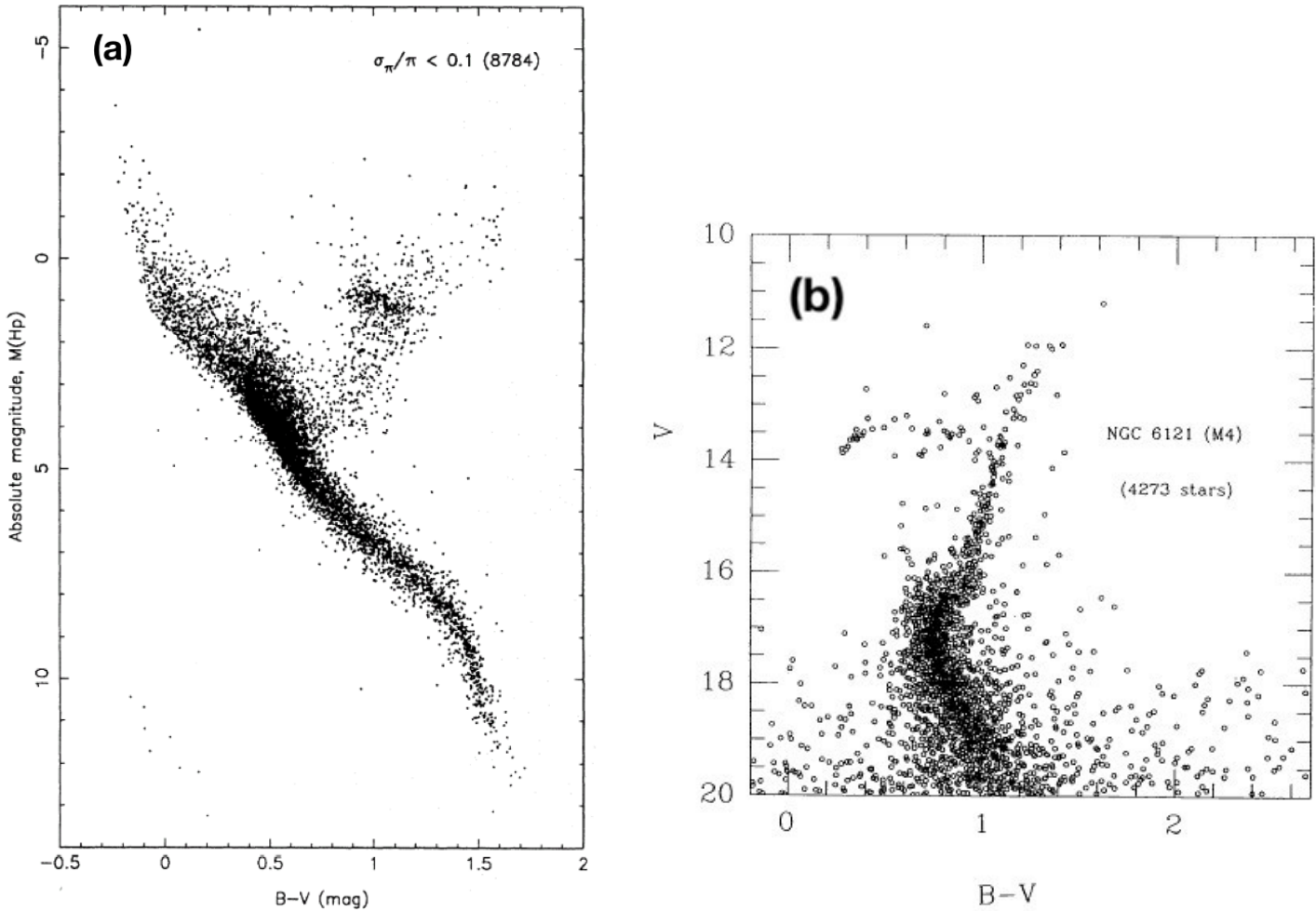


Figure 1: Color-magnitude diagrams of (a) nearby stars and (b) a globular cluster.

(c) (Difference in Perspectives)

According to the unified model of active galactic nuclei (AGN), as shown in Figure 2, the central black hole resides within a dusty torus. Different orientations of the torus with respect to the line of sight of the observer would yield different observations. The AGN is classified as a **Type I Seyfert** if the line of sight is **NOT obstructed** by the dusty torus whereas the AGN is classified as a **Type II Seyfert** if the line of sight **IS obstructed** by the dusty torus.

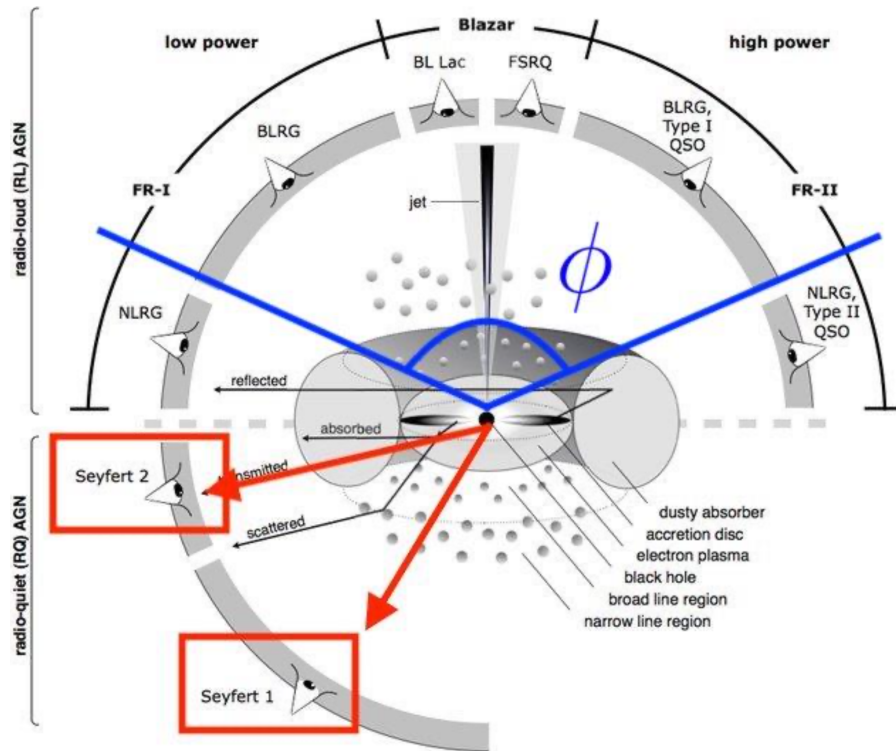


Figure 2: The unified model of Active Galactic Nuclei (AGNs). Credit: Beckmann & Shrader (2012)

The ratio of Type I to Type II Seyferts is about 1:4 in current observation samples. Assuming that the discrepancies are entirely due to the stochastic nature of the orientation, estimate the average opening angle ϕ of the dusty torus. Show the reasoning behind your estimation clearly.

2. (Watch Out! Stars!)

There are roughly 10^{11} stars in the Milky Way. Have you ever wondered, since there are so many stars, should we be worried that our Sun will “bump into” another star and the Earth will be destroyed as a result? In this task, we will attempt to answer this question.

- For simplicity, assume that the Milky Way is a sphere with 10 kpc in radius and all stars have the same mass as the Sun. Estimate, of the order of magnitude, the typical velocity of stars in the Milky Way using Virial Theorem.
- What is the total volume spanned by the Milky Way?
- Let's assume that each star has an “influence radius” of 10 au, similar to the Solar System, and each star has lived since the Big Bang (~ 10 Gyr). Estimate the maximum volume travelled by all the stars in the Milky Way.
- From your calculations above, should we lose sleep and worry that our Sun will crash into another star? Explain your reasoning.

3. (α Centauri)

α Centauri is a triple star system which consists of both the closest star and exoplanet to the Sun. The brightest star α Centauri A is located at $(\alpha_A, \delta_A) = (14^h 39.6^m, -60^\circ 50.5')$, whereas the nearest star Proxima Centauri is located at $(\alpha_{PC}, \delta_{PC}) = (14^h 29.7^m, -62^\circ 40.8')$.

- Calculate the angular separation between the two stars.
- The apparent magnitudes of the three stars are +0.1, +1.33, and +10.75.
What is the total apparent magnitude of the stellar system?

4. (Dyson Structures)

Dyson spheres are hypothetical megastructures that surround a star and transform its radiative energy outwards as a utilizable energy source for an advanced civilization.

- In some alternate universe, astrophysicists decide to contribute towards mankind's survival and plan on constructing a Dyson sphere around our Sun to solve Earth's energy crisis. The world's electrical consumption amounts to about 2.5×10^{13} kWh in a year. Calculate the minimum efficiency η of the Dyson sphere to generate sufficient power for electrical use.
- However, a monolithic rigid Dyson sphere is mechanically unstable due to the pressure and gravity of the central star. Hence, a Dyson 'swarm' consisting of individual pieces of the original sphere is more plausible. Suppose the particles of this Dyson swarm each has an effective collecting area of 100 cm^2 , how many particles are needed to achieve a sky covering fraction of 80% at a distance of 0.7 au from the Sun?
- Expanding on (b), assume that the entire megastructure has transfers energy at a 20% efficiency, what is the total efficiency of the Dyson swarm in utilizing the Sun's energy?
- Since the Dyson swarm is not 100% efficient, unused energy is re-radiated as heat, and leaves a detectable trace of advanced civilization. Suppose an alien civilization resides on Proxima Centauri b, at what wavelength would our radiation signal peak?

5. (The Temperature is Just Right)

With space satellites like the Kepler mission, we have discovered thousands of planets outside the solar system, But clearly, not all planets are as interesting as Earth. A fundamental question that humankind has asked ourselves is, "Are we alone?" A typical search strategy adopted by astronomers is focusing on planets that are located in what we call the Goldilocks Zone — i.e. planets that are not too close to their host stars (else they would be too hot to be habitable), nor too far (too cold). In this question, we will learn how we can roughly estimate the temperature of a planet from the property of its host star. In particular, we will estimate the temperature of the Earth from first principles.

- As measured from Earth, the Sun has a radiation flux of 1360 W m^{-2} .
How much energy does the Earth intersect per unit time?
- Assuming that the Earth is not accumulating energy, in this case, the energy input should be compensated by the energy output. Assuming the Stefan-Boltzmann Law for the emission, if Earth is a blackbody, estimate what would be the temperature of the Earth.
Note that the surface area of Earth emitting energy is not the same as the surface area of which Earth intercepts energy.
- What would be the temperature of Earth, if Earth is twice as large in radius?

6. (Observing Venus)

Venus is the second closest planet to the Sun, orbiting 0.7 au from the Sun. In this question, you can assume that all orbits are circular.

- Venus is best visible during greatest elongation, where the angular separation between Venus and the Sun is at its maximum. Draw a diagram with the Sun, Venus and Earth, indicating the point of greatest elongation.
- Calculate the angular separation between Venus and Earth during greatest elongation.
- Calculate the distance to Venus during greatest elongation.
- Below is an image of Venus captured by Malaysian amateur astronomer Shahrin Ahmad on June 2, 2020, near inferior conjunction. In the image, Venus spanned an angular size of $57.7''$, with only 0.1% of its area illuminated.



Figure 3: Venus 2 days before inferior conjunction. Courtesy of Shahrin Ahmad.

Assuming that the albedo of Venus does not depend on phase angle, calculate the difference of magnitude of Venus between Shahrin Ahmad's image and that during greatest elongation.

- A radio telescope operating at a frequency $\nu = 1$ GHz observed a flux density of about 2.5 Jy during inferior conjunction. What is the brightness temperature of Venus inferred?

Note that $1 \text{ Jy} = 10^{-26} \text{ W m}^{-2} \text{ Hz}^{-1}$

7. (Evolution of Supernova Explosions)

Supernova explosions are major transient events at the late evolutionary stages of stellar evolution. The extreme environments created by supernova explosions play vital roles in chemical enrichment, stellar formation, and so on. The blast waves of supernovae evolve very differently in two distinct phases: an initial energy-conserving **Sedov-Taylor expansion phase**, and a later momentum-conserving **snowplow phase**.

Note that if you are asked to give an estimate, you can express your answer as a proportionality. If you need to use the proportionality in **numerical** estimations, you are free to assume the proportionality constants are of order unity.

- (a) In the initial Sedov-Taylor expansion phase, we can approximate the expansion as adiabatic. Hence, consider an explosion that injects energy E into the ISM of uniform density ρ_0 as shown in Figure 4.

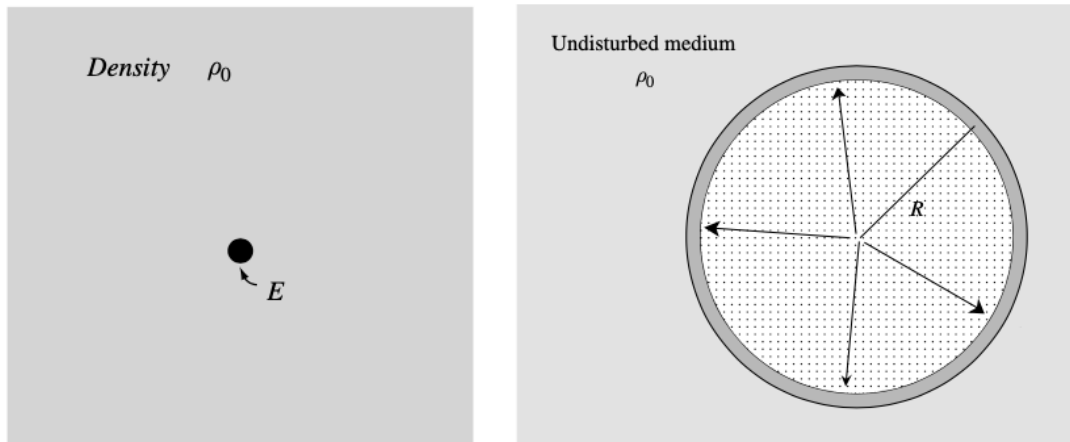


Figure 4: Supernova explosion and consequent blast wave

Estimate the radius of the shock wave R at a time t after the explosion, in terms of ρ_0 , and E .

- (b) Estimate the velocity U_{ST} of the shock front expansion in the Sedov-Taylor phase, in terms of t , ρ_0 and E .
- (c) At later stages, energy lost through radiation become substantial in the course of the expansion. The expansion goes into the momentum-conserving snowplow stage. Assume that the shock front acquires a momentum p at the transition between the Sedov-Taylor and the snowplow phase. Estimate the radius of the shock wave R at time τ after the transition, in terms of p and ρ_0 .
- (d) A typical supernova expels about $1 M_\odot$ of ejecta at a speed around 10^4 km/s, whereas the density of the ISM averages around 10^{-21} kg/m³. The average cooling function of the shock front is about 10^{-22} J m⁻³ s⁻¹. Estimate the time (**in years**) it will take for the supernova to enter the snowplow phase.
- (e) Estimate the momentum the shock front would acquire at the transition to the snowplow phase.
- (f) A spherical supernova remnant 6000 light years away is observed to span 10 arc-minutes. Assume the total energy injected by the progenitor supernova is typical. Is the supernova more likely to be in the **Sedov-Taylor phase**, or the **snowplow phase**? State your reasoning.

8. (Halley's Comet)

Halley's Comet, officially designated 1P/Halley, is a short-period comet with a period of 75.3 years. Halley's comet was last seen when it crossed its perihelion on February 9, 1986, which is 0.586 au from the Sun.

(a) (Orbit of Halley's Comet)

- i. Calculate the semi-major axis a of the orbit of Halley's comet.
- ii. Calculate the eccentricity e of the orbit of Halley's comet.
- iii. Calculate the semi-minor axis b of the orbit of Halley's comet.

(b) (Where Are You Now?)

Next, we would like to know where Halley's comet is now.

When considering the position of an object in an elliptical orbit as a function of time, there are two important angles (called 'anomalies') necessary to do the calculation, as defined in Figure 5. We do this by constructing a circular orbit concentric with the ellipse and with the same orbital period.

The eccentric anomaly E is then the angle between the major axis and the perpendicular projection of the object (some time t after crossing the perihelion) onto the circle as measured from the centre ($\angle xcz$). The mean anomaly M is the angle between the major axis and where the object would have been at time t if it was indeed on the circular orbit ($\angle ycz$ in the figure). This construction allows the two shaded triangles to have the same area.

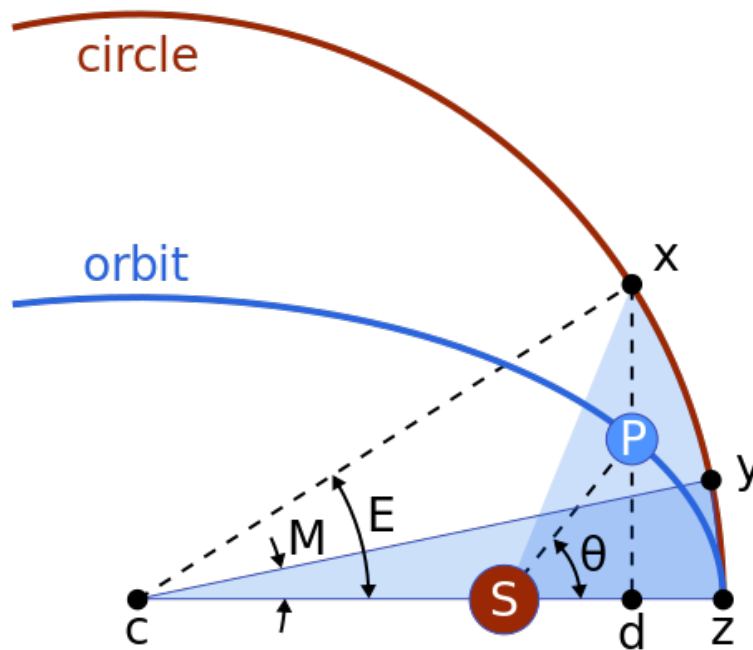


Figure 5: Definition of anomalies. The Sun is labeled S and the orbiting particle P . M and E are the mean and eccentric anomalies respectively. The angle θ is the true anomaly, measured from the focus and is not needed for this question. Courtesy of Wikipedia Commons.

- i. Let the latest crossing of perihelion occur at $t = \tau$.
Express the mean anomaly M at time t in terms of τ and the period T .
- ii. The eccentric anomaly can be related to the mean anomaly through Kepler's Equation,

$$M = E - e \sin E \quad (7)$$

Prove the Kepler's equation. You may use this equation in subsequent parts, without proving it.

- iii. Calculate the number of days t_0 since Halley's comet last cross its perihelion. Using the Kepler's equation, find the eccentric anomaly of Halley's Comet now, $E(t_0)$.
Hint: You may need to use repetitive iteration or other iteration methods such as Newton-Raphson to obtain $E(t_0)$. If you are unable to solve this part of the question, take $E(t_0) = 3$.
- iv. Derive a formula for the distance r from the focus for an elliptical orbit ($r = \overline{SP}$ in the figure) in terms of the semi-major axis a , the eccentricity e , and the eccentric anomaly E .
- v. Hence, calculate the current heliocentric distance of Halley's Comet.

(c) (Where Are You Now? – Continued)

In the latest passing of Halley's Comet, the viewing circumstances were the worst possible for observers on Earth. This was because the Earth and the comet were on opposite sides of the Sun, specifically a phase angle $\phi \approx 167^\circ$ apart during perihelion.

- i. To simplify calculations, assume very coarsely that the comet were exactly at superior conjunction (exactly behind the Sun) during its last crossing, and that its orbit is coplanar with Earth's.
Given that Halley's comet orbits prograde to Earth's rotation. Find the current phase angle $\phi(t_0)$ between Earth and Halley's comet.
- ii. Calculate the current distance to Halley's Comet from Earth.
- iii. In the 1986 crossing of Halley's Comet, it was reported to have an apparent magnitude of +2.1
Calculate the current magnitude of Halley's comet.